International Journal of Theoretical Physics, Vol. 35, No. 5, 1996

Symmetries and Constants of the Motion for Singular Lagrangian Systems

Manuel de Léon[1] and David Martín de Diego[2]

Received October 13, 1995

A classification of infinitesimal symmetries of singular autonomous and nonautonomous Lagrangian systems is obtained. The relationship between infinitesimal symmetries and constants of the motion is given.

1. INTRODUCTION

As is well known, it is important to obtain symmetries of Lagrangian systems in order to integrate the motion equations (Binz *et al.*, 1988; de León and Rodrigues, 1989; Olver, 1986; Marmo, 1988). In a recent paper (de León and Martín de Diego, 1995; see also de Léon and Martín de Diego, 1994a,b) we have classified the infinitesimal symmetries of higher order regular Lagrangian systems and established the relationship between them and the constants of the motion. In the present paper a classification of infinitesimal symmetries of presymplectic systems is given and the corresponding constants of the motion are obtained. Several Noether-type theorems are proved. The results are applied to the interesting case of singular Lagrangian systems. Our procedure is the following. First, we consider the case of presymplectic systems admitting a global dynamics. This assumption significantly simplifies the matter. Next, we consider the arbitrary case and apply the results to the final constraint submanifold, which admits a global dynamics. A similar procedure also works for the nonautonomous case and precosymplectic systems. In both cases, the Hamiltonian counterpart is studied and the results on both sides are related by means of the Legendre transformation. In a

[1] Instituto de Matemáticas y Físical Fundamental, Consejo Superior de Investigaciones Científicas, Serrano 123, 28006 Madrid, Spain; e-mail: mdeleon@pinar1.csic.es.
[2] Departamento de Economía Aplicada Cuantitativa, Facultad de Ciencias Económicas y Empresariales, UNED, 28040 Madrid, Spain; e-mail: ceedd05@cc.csic.es.

0020-7748/96/0500-0975$09.50/0

forthcoming paper we shall classify the symmetries of singular higher order Lagrangian systems.

Our results are an extension of previous ones by Crampin (1983) and Prince (1983, 1985) (see also de León and Rodrigues, 1989), and complete the results of Cariñena and Rañada (1988), Marmo *et al.* (1983), and Ferrario and Passerini (1990). The time-dependent case was studied by Cariñena and Fernández (1993) and Cariñena *et al.* (1991, 1992) by using the technique of sections along maps. Our approach is consistent with the one by Prince and uses the cosymplectic formalism developed in de León and Rodrigues (1988, 1990).

The paper is organized as follows. In Section 2, we recall the constraint algorithm developed by Gotay and Nester (1979, 1980; Gotay, 1979). Section 3 is devoted to a study of infinitesimal symmetries of presymplectic systems with a global dynamics. In Section 4, we extend these results to a general presymplectic system. The classification of infinitesimal symmetries is given in Section 5 and the corresponding constants of the motion for degenerate Lagrangian systems are obtained. In Section 6, the relationship between the Lagrangian and the Hamiltonian formalisms is studied. The second-order problem is considered in Section 7. Sections 8–11 apply these results to the following particular cases: generalized Hamiltonian dynamics, affine Lagrangians on the velocities, degenerate Lagrangian systems of type II, and degenerate Lagrangian systems with a Lie group of symmetries. The case of nonautonomous Lagrangian systems is studied in Section 12 as an application of the results for arbitrary precosymplectic systems.

2. THE CONSTRAINT ALGORITHM

Let M be an n-dimensional manifold, ω a closed 2-form with constant rank, and α a closed 1-form. The triple (M, ω, α) is said to be a *presymplectic system.*

The dynamics is determined by the solutions of the equation

$$i_X\omega = \alpha \tag{1}$$

Since ω is not symplectic, (1) has no solution, in general, and even if it exists it will not be unique. Let b: $TM \to T^*M$ be the map defined by $b(X) = i_X\omega$. It may happen that b is not surjective. We denote by ker ω the kernel of b, i.e., ker b = ker ω.

Gotay (1979) and Gotay and Nester (1979) developed a constraint algorithm for presymplectic systems. They consider the points of M where (1) has a solution and suppose that this set M_2 is a submanifold of M. Nevertheless, these solutions on M_2 may not be tangent to M_2. Then, we have to restrict

M_2 to a submanifold where the solutions of (1) are tangent to M_2. Proceeding further, we obtain a sequence of submanifolds:

$$\cdots \to M_k \to \cdots \to M_2 \to M_1 = M$$

Alternatively, these constraint submanifolds may be described as follows:

$$M_i = \{x \in M/\alpha(x)(v) = 0, \forall v \in T_x M_{i-1}^{\perp}\}$$

where

$$T_x M_{i-1}^{\perp} = \{v \in T_x M/\omega(x)(u, v) = 0, \forall u \in T_x M_{i-1}\}$$

We call M_2 the *secondary constraint submanifold*, M_3 the *tertiary constraint submanifold*, and, in general, M_i is the *i-ary constraint submanifold.*

If the algorithm stabilizes, i.e., there exists a positive integer $k \in \mathbb{N}$ such that $M_k = M_{k+1}$ and $\dim M_k \neq 0$, then we have a *final constraint submanifold* $M_f = M_k$, on which a vector field X exists such that

$$(i_X\omega = \alpha)_{/M_f} \tag{2}$$

If ξ is a solution of (2), then every arbitrary solution on M_f is of the form $\xi' = \xi + Y$, where $Y \in (\ker \omega \cap TM_f)$.

3. SYMMETRIES AND CONSTANTS OF THE MOTION FOR A PRESYMPLECTIC SYSTEM WITH A GLOBAL DYNAMICS

In this section, we give a classification of symmetries and constants of the motion for a particular case of presymplectic systems, those which admit a global dynamics (Cariñena and Rañada, 1988).

We say that a presymplectic system (M, ω, α) admits a global dynamics if there exists a vector field ξ on M such that ξ satisfies (1). This condition is equivalent to the following one:

$$\alpha(\ker \omega)(x) = 0, \qquad \forall x \in M$$

Definition 3.1. A function $F: M \to \mathbb{R}$ is said to be a *constant of the motion* of ξ if $\xi F = 0$.

Thus, if γ is an integral curve of ξ, then $F \circ \gamma$ is a constant function.

Definition 3.2. A diffeomorphism $\phi: M \to M$ is said to be a *symmetry* of ξ if ϕ maps integral curves of ξ onto integral curves of ξ, i.e., $T\phi(\xi) = \xi$.

Definition 3.3. A *dynamical symmetry* of ξ is a vector field X on M such that its flow consists of symmetries of ξ, or, equivalently, $[X, \xi] = 0$.

We denote by $\mathfrak{X}^{\omega}(M)$ the set of all the solutions of (1):

$$\mathfrak{X}^{\omega}(M) = \{X \in \mathfrak{X}(M)/i_X\omega = \alpha\}$$

Definition 3.4. A function F: $M \to \mathbb{R}$ is said to be a *constant of the motion* of $\mathfrak{X}^{\omega}(M)$ if F is constant along all the integral curves of any solution of (1). That is, if F satisfies

$$\mathfrak{X}^{\omega}(M)F = 0$$

Therefore, if F is a constant of the motion of $\mathfrak{X}^{\omega}(M)$, we have

$$(\ker \omega)F = 0$$

Definition 3.5. A diffeomorphism ϕ: $M \to M$ is said to be a *symmetry* of $\mathfrak{X}^{\omega}(M)$ if ϕ satisfies

$$T\phi(\xi) \in \mathfrak{X}^{\omega}(M)$$

for all $\xi \in \mathfrak{X}^{\omega}(M)$.

Definition 3.6. A *dynamical symmetry* of $\mathfrak{X}^{\omega}(M)$ is a vector field such that

$$[X, \mathfrak{X}^{\omega}(M)] \subset \ker \omega$$

Remark 3.1. If the foliation defined by ker ω is a fibration, then the quotient manifold $\tilde{M} = M/\ker \omega$ admits a structure of differentiable manifold and the canonical projection π: $M \to \tilde{M}$ is a surjective submersion. In that case, there exists a unique symplectic form $\tilde{\omega}$ on $\tilde{M}$ such that $\pi^*\tilde{\omega} = \omega$. Since we have supposed that the presymplectic system admits a global dynamics, the 1-form α projects onto a 1-form $\tilde{\alpha}$ on $\tilde{M}$ such that $\pi^*\tilde{\alpha} = \alpha$. Since $\tilde{\omega}$ is symplectic, there exists a unique vector field $\tilde{\xi}$ on $\tilde{M}$ such that

$$i_{\tilde{\xi}}\tilde{\omega} = \tilde{\alpha}$$

It is easy to prove that all the solutions of (1) are projectable and, in fact, all them project onto the vector field $\tilde{\xi}$. From Definition 3.6, we deduce that

$$[X, \ker \omega] \subset \ker \omega$$

Therefore, X projects onto a vector field $\tilde{X}$ on M such that

$$[\tilde{X}, \tilde{\xi}] = 0$$

In other words, $\tilde{X}$ is a dynamical symmetry of $\tilde{\xi}$. This fact justifies Definition 3.6.

Remark 3.2. If F is a constant of the motion of $\mathfrak{X}^{\omega}(M)$, then XF is also a constant of the motion of $\mathfrak{X}^{\omega}(M)$. In fact, since $(\ker \omega)F = 0$, we have

$$[X, \xi]F = X(\xi F) - \xi(XF) = -\xi(XF) = 0$$

for all $\xi \in \mathfrak{X}^{\omega}(M)$.

We denote by $D(\mathfrak{X}^{\omega}(M))$ the set of dynamical symmetries of $\mathfrak{X}^{\omega}(M)$. Let X and Y be two dynamical symmetries of $\mathfrak{X}^{\omega}(M)$. Then $[X, Y]$ is also a dynamical symmetry of $\mathfrak{X}^{\omega}(M)$. In fact,

$$\begin{aligned}[][[X, Y], \mathfrak{X}^{\omega}(M)] &= [X, [Y, \mathfrak{X}^{\omega}(M)]] + [Y, [\mathfrak{X}^{\omega}(M), X]] \\ &\subset [X, \ker \omega] + [Y, \ker \omega] \subset \ker \omega\end{aligned}$$

Therefore, $D(\mathfrak{X}^{\omega}(M))$ is a Lie subalgebra of the Lie algebra $\mathfrak{X}(M)$ of vector fields on M.

Definition 3.7. A *Cartan symmetry* of (M, ω, α) is a vector field X on M such that:

1. $i_X\omega = dG$, for some function $G: M \to \mathbb{R}$.
2. $i_X\alpha = 0$.

Proposition 3.1. If X is a Cartan symmetry of (M, ω, α) then X is a dynamical symmetry of $\mathfrak{X}^{\omega}(M)$.

Proof. If X is a Cartan symmetry, then, for each solution ξ of (1) we have

$$\begin{aligned} i_{[X,\xi]}\omega &= L_X i_\xi \omega - i_\xi L_X \omega \\ &= L_X\alpha = d(i_X\alpha) = 0 \end{aligned}$$

Thus, $[X, \xi] \in \ker \omega$, and therefore X is a dynamical symmetry of $\mathfrak{X}^{\omega}(M)$. ∎

Let $C(\omega, \alpha)$ be the set of all Cartan symmetries of (M, ω, α). From a straightforward computation, we deduce that, if X and Y are Cartan symmetries of (M, ω, α), $[X, Y]$ is also a Cartan symmetry. Therefore, $C(\omega, \alpha)$ is a Lie subalgebra of $\mathfrak{X}(M)$. From Proposition 3.1 we obtain that

$$C(\omega, \alpha) \subset D(\mathfrak{X}^{\omega}(M))$$

Theorem 3.1 (Noether Theorem). If X is a Cartan symmetry of (M, ω, α), then the function G (as in Definition 3.7) is a constant of the motion of $\mathfrak{X}^{\omega}(M)$. Conversely, if G is a constant of the motion of $\mathfrak{X}^{\omega}(M)$, then there exists a vector field X such that

$$i_X\omega = dG$$

and, moreover, X is a Cartan symmetry of (M, ω, α), and every vector field $X + Z$ with $Z \in \ker \omega$ is also a Cartan symmetry of (M, ω, α).

Proof. In fact, if G is a constant of the motion of $\mathfrak{X}^{\omega}(M)$, it satisfies $(\ker \omega)G = 0$. Thus, the equation

$$i_Y \omega = dG$$

has a globally defined solution X on M and, since

$$0 = \xi G = i_\xi \, dG = i_\xi i_X \omega$$
$$= -i_X i_\xi \omega = -i_X \alpha$$

we deduce that X is a Cartan symmetry of (M, ω, α) and that every vector field $X + Z$ with $Z \in \ker \omega$ is a Cartan symmetry.

Conversely, if X is a Cartan symmetry of (M, ω, α), then, for each solution ξ of (1), we obtain

$$0 = i_X \alpha = i_X i_\xi \omega = -i_\xi i_X \omega = -\xi(G)$$

Therefore, G is a constant of the motion of $\mathfrak{X}^{\omega}(M)$. ■

4. SYMMETRIES AND CONSTANTS OF THE MOTION FOR GENERAL PRESYMPLECTIC SYSTEMS

Let (M, ω, α) be a presymplectic system. In general, (1) does not have a globally defined solution as in Section 3. The constraint algorithm allows us to obtain (if it is possible) a final constraint submanifold M_f.

First of all, consider the presymplectic structure $(M_f, j_f^* \omega, j_f^* \alpha)$ where j_f: $M_f \to M$ is the embedding of M_f into M. We know that any solution of (1) is a vector field X on M_f such that

$$(i_X \omega = \alpha)_{/M_f} \tag{3}$$

If we put $\omega_{M_f} = j_f^* \omega$ and $\alpha_{M_f} = j_f^* \alpha$, it is easy to prove that, if ξ is a solution of (1), then ξ is also a solution of the following equation:

$$i_X \omega_{M_f} = \alpha_{M_f} \tag{4}$$

Define the sets

$$\mathfrak{X}^{\omega_{M_f}}(M_f) = \{X \in \mathfrak{X}(M_f)/i_X \omega_{M_f} = \alpha_{M_f}\}$$
$$\mathfrak{X}^{\omega}(M_f) = \{X \in \mathfrak{X}(M_f)/(i_X \omega = \alpha)_{/M_f}\}$$

Hence, $\mathfrak{X}^{\omega}(M_f) \subset \mathfrak{X}^{\omega_{M_f}}(M_f)$.

If we suppose that the rank of ω_{M_f} is constant, then $(M_f, \omega_{M_f}, \alpha_{M_f})$ is a presymplectic system with a global dynamics. We can therefore apply all the definitions and results of Section 3 to this presymplectic system.

We obtain that $D(\mathfrak{X}^{\omega_{M_f}}(M_f))$ is a Lie subalgebra of $\mathfrak{X}(M_f)$ and $C(\omega_{M_f}, \alpha_{M_f})$ is a Lie subalgebra of $\mathfrak{X}(M_f)$. Since any Cartan symmetry of $(M, \omega_{M_f}, \alpha_{M_f})$ is a dynamical symmetry of $\mathfrak{X}^{\omega_{M_f}}(M_f)$, we deduce that

$$C(\omega_{M_f}, \alpha_{M_f}) \subset D(\mathfrak{X}^{\omega_{M_f}}(M_f))$$

Theorem 3.1 now reads as follows:

Theorem 4.1 (Noether Theorem). If X is a Cartan symmetry of $(M_f, \omega_{M_f}, \alpha_{M_f})$, then G is a constant of the motion of $\mathfrak{X}^{\omega_{M_f}}(M_f)$. Conversely, if G is a constant of the motion of $\mathfrak{X}^{\omega_{M_f}}(M_f)$, then there exists a vector field X on M_f such that

$$i_X \omega_{M_f} = dG$$

and, moreover, X is a Cartan symmetry and every vector field $X + Z$, with $Z \in \ker \omega_{M_f}$, is a Cartan symmetry, too.

Since $\mathfrak{X}^{\omega}(M_f) \subset \mathfrak{X}^{\omega_{M_f}}(M_f)$, we can distinguish another type of symmetry and constant of the motion.

Definition 4.1. 1. A function F: $M_f \to \mathrm{R}$ is said to be a *constant of the motion* of $\mathfrak{X}^{\omega}(M_f)$ if F is a constant along all the integral curves of the solutions of (3), i.e.,

$$\mathfrak{X}^{\omega}(M_f)F = 0$$

2. A diffeomorphism ϕ: $M_f \to M_f$ is said to be a *symmetry* of $\mathfrak{X}^{\omega}(M_f)$ if ϕ maps integral curves of solutions of (3) onto integral curves of solutions of (3).

3. A *dynamical symmetry* of $\mathfrak{X}^{\omega}(M_f)$ is a vector field on M_f such that

$$[X, \mathfrak{X}^{\omega}(M_f)] \in \ker \omega \cap TM_f$$

We now consider diffeomorphisms ϕ: $M \to M$ such that they preserve the 2-form ω and the 1-form α (i.e., they preserve the presymplectic structure):

$$\phi^*\omega = \omega, \qquad \phi^*\alpha = \alpha$$

Proposition 4.1. If the diffeomorphism ϕ: $M \to M$ preserves the presymplectic structure, then it restricts to a diffeomorphism ϕ_i: $M_i \to M_i$, where M_i is the i-ary constraint submanifold. Therefore, ϕ restricts to a diffeomorphism ϕ_f: $M_f \to M_f$.

Proof. If $i = 1$, the proposition is trivially true. Now, suppose that the proposition is true for $i = m$ and we shall prove that it is also true for $i = m + 1$.

We shall prove that, if $v \in T_x M_m^\perp$, then $d\phi(x)(v) \in T_{\phi(x)} M_m^\perp$. In fact, for each $u \in T_{\phi(x)} M_m$, we obtain

$$\omega(\phi(x))(d\phi(x)(v), u) = \omega(x)(v, d\phi^{-1}(\phi(x))u) = 0$$

because ω is ϕ-invariant and $d\phi^{-1}(\phi(x))u \in T_x M_m$ by the hypothesis of induction. Thus, we deduce that

$$d\phi(x)(T_x M_m^\perp) = T_{\phi(x)} M_m^\perp$$

We now only have to prove that, if $x \in M_{m+1}$, then $\phi(x) \in M_{m+1}$, i.e., for all $v \in T_{\phi(x)} M_m^\perp$, $\alpha(\phi(x))(v) = 0$. But, since α is also ϕ-invariant, we obtain

$$\alpha(\phi(x))(v) = \alpha(x)(d\phi^{-1}(\phi(x))v) = 0 \quad \blacksquare$$

Corollary 4.1. Let X be a vector field on M such that:
1. $i_X\omega = dG$, for some function $G: M \to \mathbb{R}$.
2. $i_X\alpha = 0$.

Then $X_{/M_f}$ is a Cartan symmetry of $(M_f, \omega_{M_f}, \alpha_{M_f})$.

Proof. Since the flow of X consists of diffeomorphisms which preserve the presymplectic structure, then, from Proposition 4.1, X is tangent to M_f. Moreover, since X satisfies

$$i_X\omega = dG$$

the restriction of X to M_f also satisfies

$$i_{X_{/M_f}}\omega_{M_f} = d(G_{/M_f})$$

Finally,

$$i_{X_{/M_f}}\alpha_{M_f} = 0$$

Thus, $X_{/M_f}$ is a Cartan symmetry of $(M_f, \omega_{M_f}, \alpha_{M_f})$ and $G_{/M_f}$ is a constant of the motion of $\mathfrak{X}^{\omega_{M_f}}(M_f)$. $\blacksquare$

Example 4.1. Consider the presymplectic system $(\mathbb{R}^6, \omega, \alpha)$, where

$$\omega = dx_1 \wedge dx_4 - dx_2 \wedge dx_3$$

$$\alpha = x_4\, dx_4 - x_3\, dx_5 - x_5\, dx_3$$

with $(x^1, x^2, x^3, x^4, x^5, x^6)$ the standard coordinates on $\mathbb{R}^6$. It is easy to prove that ker ω is generated by $\partial/\partial x_5$ and $\partial/\partial x_6$. The only secondary constraint is $\phi_1 = x_3 = 0$. Since there are not tertiary constraints, the constraint algorithm ends in M_2, i.e.,

$$M_f = M_2 = \{(x_1, x_2, x_3, x_4, x_5, x_6) \in \mathbb{R}^6 / x_3 = 0\}$$

The solutions of the equation

$$(i_X \omega = \alpha)_{/M_f}$$

are

$$\mathfrak{X}^\omega(M_f) = x_4 \frac{\partial}{\partial x_1} + \ker \omega$$

If we denote by j: $M_f \to \mathbb{R}^6$ the embedding of M_f in $\mathbb{R}^6$, then

$$j^*\omega = \omega_{M_f} = dx_1 \wedge dx_4$$

Therefore, $\ker \omega_{M_f}$ is generated by $\partial/\partial x_2$, $\partial/\partial x_5$, and $\partial/\partial x_6$. The solutions of the equation

$$i_X \omega_{M_f} = j^*\alpha$$

are

$$\mathfrak{X}^{\omega_{M_f}}(M_f) = x_4 \frac{\partial}{\partial x_1} + \ker \omega_{M_f}$$

Thus, $\mathfrak{X}^\omega(M_f)$ is strictly contained in $\mathfrak{X}^{\omega_{M_f}}(M_f)$. We shall now study the symmetries and constants of the motion for the presymplectic system (M, ω, α).

A function F: $M_f \to \mathbb{R}$ is a constant of the motion of $\mathfrak{X}^\omega(M_f)$ if it satisfies the following conditions:

$$x_4 \frac{\partial F}{\partial x_1} = 0, \qquad \frac{\partial F}{\partial x_5} = 0, \qquad \frac{\partial F}{\partial x_6} = 0$$

Therefore, each function F which depends only on x_2 and x_4 is a constant of the motion of $\mathfrak{X}^\omega(M_f)$. For instance, $F_1(x_1, x_2, x_4, x_5, x_6) = x_4$ and $F_2(x_1, x_2, x_4, x_5, x_6) = x_2$ are constants of the motion.

A function F: $M_f \to \mathbb{R}$ is a constant of the motion of $\mathfrak{X}^{\omega_{M_f}}(M_f)$ if it satisfies

$$x_4 \frac{\partial F}{\partial x_1} = 0, \qquad \frac{\partial F}{\partial x_2} = 0, \qquad \frac{\partial F}{\partial x_5} = 0, \qquad \frac{\partial F}{\partial x_6} = 0$$

The functions F which are constants of the motion of $\mathfrak{X}^{\omega_{M_f}}(M_f)$ are the ones which depend only of x_4, for instance

$$F_1(x_1, x_2, x_4, x_5, x_6) = x_4$$

Obviously, all the constants of the motion of $\mathfrak{X}^{\omega_{M_f}}(M_f)$ are also constants of the motion of $\mathfrak{X}^\omega(M_f)$.

The vector field $X = \partial/\partial x_1$ on R^6 satisfies

$$i_X\omega = dG, \qquad \text{where} \quad G(x_1, x_2, x_3, x_4, x_5, x_6) = x_4, \quad i_X\alpha = 0$$

From Corollary 4.1, we deduce that X is a Cartan symmetry of $(M_f, \omega_{M_f}, \alpha_{M_f})$ and $G_{/M_f}$ is a constant of the motion of $\mathfrak{X}^{\omega_{M_f}}(M_f)$.

5. SYMMETRIES AND CONSTANTS OF THE MOTION FOR SINGULAR LAGRANGIAN SYSTEMS

Let Q be an n-dimensional differentiable manifold. Consider a Lagrangian $L: TQ \to \mathbb{R}$ such that the Hessian matrix

$$\left(\frac{\partial^2 L}{\partial \dot{q}^A \partial \dot{q}^B}\right)$$

is nonregular. This type of Lagrangian is called *singular* or *degenerate*. Let E_L be the energy associated with L, defined by $E_L = CL - L$, where C is the Liouville vector field on TQ. We denote by α_L the Poincaré–Cartan 1-form defined by $\alpha_L = J^*(dL)$ and, by ω_L the Poincaré–Cartan 2-form defined by $\omega_L = -d\alpha_L$, where J is the canonical almost tangent structure on TQ. Let us recall that J is a (1,1) tensor field on TQ locally defined by

$$J = \frac{\partial}{\partial \dot{q}^A} \otimes dq^A$$

and C is the infinitesimal generator of the dilations on TQ:

$$C = \dot{q}^A \frac{\partial}{\partial \dot{q}^A}$$

where $(q^A, \dot{q}^A)$ are fibered coordinates on TQ. Thus, we have

$$E_L = \sum_{A=1}^{n} \dot{q}^A \frac{\partial L}{\partial \dot{q}^A} - L(q^A, \dot{q}^A)$$

$$\alpha_L = \sum_{A=1}^{n} \frac{\partial L}{\partial \dot{q}^A} dq^A$$

$$\omega_L = \frac{\partial^2 L}{\partial \dot{q}^A \partial q^B} dq^A \wedge dq^B + \frac{\partial^2 L}{\partial \dot{q}^A \partial \dot{q}^B} dq^A \wedge d\dot{q}^B$$

We suppose that the 2-form ω_L has constant rank and we apply the constraint algorithm to the presymplectic system (TQ, ω_L, dE_L). Then we obtain the following sequence of constraint submanifolds:

$$\cdots \rightarrow P_k \rightarrow \cdots \rightarrow P_3 \rightarrow P_2 \rightarrow P_1 = TQ$$

If the algorithm stabilizes, then there exists an integer k such that $P_{k+1} = P_k = P_f$ and P_f is the final constraint submanifold. Thus, we can translate all the definitions and results of Section 4 to this particular case.

Denote by X^c the complete lift and by X^v the vertical lift to TQ of a vector field X on Q.

Definition 5.1. A vector field X on Q is said to be a *Lie symmetry* of $\mathfrak{X}^{\omega_{P_f}}(P_f)$ if:

1. X^c is tangent to P_f.
2. $[X^c_{/P_f}, \mathfrak{X}^{\omega_{P_f}}(P_f)] \subset \ker \omega_{P_f}$

The set $\mathcal{L}(\mathfrak{X}^{\omega_{P_f}}(P_f))$ consisting of all the Lie symmetries of $\mathfrak{X}^{\omega_{P_f}}(P_f)$ is a Lie subalgebra of $\mathfrak{X}(Q)$. Moreover, we have

$$(\mathcal{L}(\mathfrak{X}^{\omega_{P_f}}(P_f)))^c_{/P_f} \subset D(\mathfrak{X}^{\omega_{P_f}}(P_f))$$

Definition 5.2. A diffeomorphism Φ: $Q \rightarrow Q$ is said to be a *symmetry* of L if $L \circ T\Phi = L$. A vector field X on Q is said to be an *infinitesimal symmetry* of L if

$$X^c L = 0$$

i.e., if its flow consists of symmetries of L.

If we denote by $I(L)$ the set of all the infinitesimal symmetries of L, then $I(L)$ is a Lie subalgebra of $\mathfrak{X}(Q)$.

Proposition 5.1. If X is an infinitesimal symmetry of L, then $\theta(X^c)_{/P_f} = (X^v L)_{/P_f}$ is a constant of the motion of $\mathfrak{X}^{\omega_{P_f}}(P_f)$.

Proof. In local coordinates, we have

$$X^c L = X^A \frac{\partial L}{\partial q^A} + \dot{q}^B \frac{\partial X^A}{\partial q^B} \frac{\partial L}{\partial \dot{q}^A} = 0$$

and

$$\begin{aligned} L_{X^c}\alpha_L &= X^c\left(\frac{\partial L}{\partial \dot{q}^B}\right) dq^B + \frac{\partial L}{\partial \dot{q}^A} \frac{\partial X^A}{\partial q^B} dq^B \\ &= \left(X^A \frac{\partial^2 L}{\partial q^A \partial \dot{q}^B} + \dot{q}^B \frac{\partial X^A}{\partial q^B} \frac{\partial^2 L}{\partial \dot{q}^A \partial \dot{q}^B} + \frac{\partial L}{\partial \dot{q}^A} \frac{\partial X^A}{\partial q^B}\right) dq^B \\ &= \left(\frac{\partial}{\partial \dot{q}^B} (X^c L)\right) dq^B \\ &= 0 \end{aligned}$$

But, if $L_{X^c}\alpha_L = 0$, then

$$i_{X^c}\omega_L = d(\alpha_L(X^c))$$

Moreover, we have $X^c E_L = 0$. We obtain the required result after applying Corollary 4.1. ■

Proceeding as in the proof of Proposition 5.1, we deduce that $I(L) \subset \mathcal{L}(\mathfrak{X}^{\omega_{P_f}}(P_f))$.

Definition 5.3. A vector field X on Q is said to be a *Noether symmetry* if

$$X^c L = G^c$$

for some function G on Q, where G^c denotes the complete lift of G to TQ.

By a similar procedure to that used in Proposition 5.1, we can characterize a Noether symmetry as follows:

1. $i_{X^c}\omega_L = dF$ for some function F.
2. $X^c E_L = 0$.

In fact, we can choose $F = \alpha_L(X^c) - G^v$, where G^v denotes the vertical lift of G.

Proposition 5.2. If X is a Noether symmetry, then $\alpha_L(X^c) - G^v$ is a constant of the motion of $\mathfrak{X}^{\omega_{P_f}}(P_f)$.

Proof. See the proof of Proposition 5.1. ■

Denote by $N(L)$ the set of all the Noether symmetries. We deduce that $N(L)$ is a Lie subalgebra of $\mathfrak{X}(Q)$ and, we have

$$I(L) \subset N(L) \subset \mathcal{L}(\mathfrak{X}^{\omega_{P_f}}(P_f))$$

$$(N(L))^c_{/P_f} \subset C(\omega_{P_f}, \alpha_{P_f})$$

6. THE RELATIONSHIP WITH THE HAMILTONIAN FORMULATION

Let $L: TQ \to R$ be an arbitrary Lagrangian. The *Legendre map* $Leg: TQ \to T^*Q$ is locally written as

$$Leg: (q^A, \dot{q}^A) \rightsquigarrow (q^A, p_A)$$

with $p_A = \partial L/\partial \dot{q}^A$. If L is singular, Leg is not a diffeomorphism. However, we suppose that L is *almost regular*, i.e., $M_1 = Leg(TQ)$ is a submanifold of T^*Q and Leg is a submersion onto M_1 with connected fibers. The submanifold M_1 will be called the *primary constraint submanifold.*

Let λ_Q be the Liouville 1-form and $\omega_Q = -d\lambda_Q$ the canonical symplectic form on T^*Q.

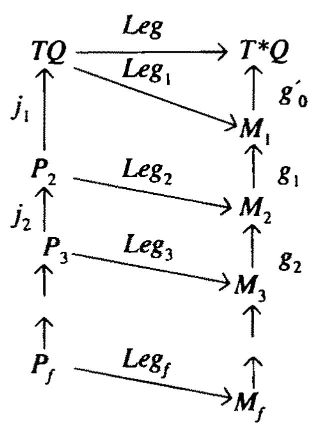
Since the Lagrangian is almost regular, the energy E_L is constant along the fibers of Leg. Therefore, E_L projects onto a function H on M_1:

$$H(Leg(x)) = E_L(x), \qquad \forall x \in TQ$$

If we denote by g_0': $M_1 \rightarrow T^*Q$ the embedding of M_1 into T^*Q, then we obtain a presymplectic system $(M_1, (g_0')^*\omega_Q, dH)$. If we now apply the constraint algorithm to it, we shall obtain the following sequence of constraint submanifolds:

$$\cdots \rightarrow M_k \rightarrow \cdots \rightarrow M_2 \rightarrow M_1$$

Denote by M_f the final constraint submanifold (if it exists) for this presymplectic system. The Legendre map restricts to each submanifold P_i, $i \geq 1$, of TQ and then we obtain a family of surjective submersions Leg_i: $P_i \rightarrow M_i$ which relates the constraint submanifolds P_i and M_i. In fact, Leg_i is a fibration, for all i. Moreover, the quotient manifold $P_i/\ker Leg_{/P_i}$ is diffeomorphic to M_i. The following commutative diagram illustrates this point:



Consider the equations

$$(i_X\omega_L = dE_L)_{/P_f} \tag{5}$$

and

$$(i_X\omega_1 = dH)_{/M_f} \tag{6}$$

Gotay and Nester (1979) proved that the Lagrangian and Hamiltonian formulations are equivalent in the following sense. Given a vector field $\xi \in$

$\mathfrak{X}(P_f)$ which is a solution of (5) and Leg_f-projectable, then its projection $Z = T\,Leg_f(\xi)$ is a solution of (6). Conversely, if $Z \in \mathfrak{X}(M_f)$ is a solution of (6), then each projectable vector field on P_f onto Z is a solution of (5).

Let $j_{i-1}: P_i \to TQ$ and $g_{i-1}: M_i \to M_1$, $1 \leq i$ (where g_0 is the identity), be the natural embeddings, and $g'_{i-1} = g'_0 \circ g_{i-1}$, $1 \leq i$. Denote by

$$\omega_{P_i} = j_i^* \omega_L, \qquad 1 \leq i \leq k$$
$$\omega_{M_i} = (g'_i)^*(\omega_Q), \qquad 1 \leq i \leq k$$

the restrictions of ω_L and ω_Q to P_i and M_i, respectively. It is easy to prove that

$$\omega_{P_i} = Leg_i^* \omega_{M_i} \qquad \text{and} \qquad j_i^* E_L = (Leg_i)^* g_i^* H$$

Proposition 6.1. If F is a constant of the motion of $\mathfrak{X}^{\omega_{P_f}}(P_f)$, then F is projectable onto M_f and its projection $\tilde{F}$ is a constant of the motion of $\mathfrak{X}^{\omega_{M_f}}(M_f)$.

Proof. In fact, if F is a constant of the motion of $\mathfrak{X}^{\omega_{P_f}}(P_f)$, then

$$(\ker \omega_{P_f})F = 0$$

But, since $\ker \omega_L \cap TP_f \subset \ker \omega_{P_f}$, we deduce that $(\ker \omega_L \cap TP_f)F = 0$. Now, since $\ker TLeg_f \subset \ker \omega_L \cap TP_f$, F is projectable. If $Z_{\omega_{M_f}}$ is a solution of the equation

$$i_X \omega_{M_f} = g_f^*(dH)$$

then any projectable vector field ξ on P_f onto $Z_{\omega_{M_f}}$ is a solution of the equation

$$(i_X \omega_{P_f} = dj_f^* E_L)_{/P_f} \tag{7}$$

since

$$\begin{aligned} i_\xi \omega_{P_f} - j_f^*(dE_L) &= i_\xi(Leg_f^* \omega_{M_f}) - Leg_f^* g_f^*(dH) \\ &= Leg_f^*(i_{Z_{\omega_{M_f}}} \omega_{M_f} - g_f^*(dH)) \\ &= 0 \end{aligned}$$

Hence, ξ is a solution of (7). Since $\xi F = 0$, we have $Z_{\omega_{M_f}}(\tilde{F}) = 0$. ■

Proposition 6.2. If F is a constant of the motion of $\mathfrak{X}^{\omega_L}(P_f)$, then F is projectable onto M_f and its projection $\tilde{F}$ is a constant of the motion of $\mathfrak{X}^{\omega_{M_1}}(M_f)$.

Proof. In fact, if $Z_{\omega_{M_1}}$ is a solution of the equation

$$(i_X \omega_{M_1} = dH)_{/M_f}$$

then any vector field ξ on P_f projectable onto $Z_{\omega_{M_1}}$ is a solution of (5). Thus, since $\xi F = 0$, we have $Z_{\omega_{M_1}} \tilde{F} = 0$. ∎

Proposition 6.3. If X is a Cartan symmetry of $(M_f, \omega_{M_f}, g_f^*(dH))$, then any vector field X' on P_f such that $TLeg_f(X') = X$ is a Cartan symmetry of $(P_f, \omega_{P_f}, j_f^*(dE_L))$.

Proof. If a vector field X on M_f satisfies (1) $i_X\omega_{M_f} = dG$ with $G: M_f \to \mathbb{R}$, and (2) $X(H_{/M_f}) = 0$, then for any $X' \in \mathfrak{X}(P_f)$ with $TLeg_f(X') = X$ we have (1) $i_{X'}\omega_{P_f} = dG'$ with $G' = Leg_f^* G$, and (2) $X'(E_{L/P_f}) = 0$. Therefore, X' is a Cartan symmetry of $(P_f, \omega_{P_f}, j_f^*(dE_L))$. ∎

Proposition 6.4. If F is a constant of the motion of $Z_{\omega_{M_1}}$ which is a solution of the equation

$$(i_X\omega_{M_1} = dH)_{/M_f}$$

then $(Leg_f)^*F$ is a constant of the motion for any ξ_{ω_L} such that $TLeg_f(\xi_{\omega_L}) = Z_{\omega_{M_1}}$.

If F is a constant of the motion of $Z_{\omega_{M_f}}$ which is a solution of the equation

$$i_X\omega_{M_f} = g_f^*(dH)$$

then $(Leg_f)^*F$ is a constant of the motion for any $\xi_{\omega_{P_f}}$ such that $TLeg_{P_f}(\xi_{\omega_{P_f}}) = Z_{\omega_{M_f}}$.

Proof. It directly follows from the equivalence of the Lagrangian and Hamiltonian formulations. ∎

Let ι be the operator which maps vector fields on Q into functions on T^*Q:

$$(\iota X)(\alpha) = \alpha(X(x))$$

for all $\alpha \in T_x^*Q$. Locally, if $X = X^A\, \partial/\partial q^A$, we get

$$(\iota X)(q^A, p_A) = p_A X^A$$

where (q^A, p_A) are the induced coordinates on T^*Q.

Let us recall that the *complete lift* of a vector field X on Q to T^*Q is the vector field X^{c*} defined by

$$i_{X^{c*}}\omega_Q = d(\iota X) \tag{8}$$

or, in other words, X^{c*} is the Hamiltonian vector field with Hamiltonian function ιX. Locally, we obtain

$$X^{c*} = X^A \frac{\partial}{\partial q^A} - p_B \frac{\partial X^B}{\partial q^A} \frac{\partial}{\partial p_A}$$

(see de León and Rodrigues, 1989).

Proposition 6.5. Let X be a vector field on M such that $X^c L = 0$, i.e., X is an infinitesimal symmetry of L. Then X^c is Leg_1-projectable and its projection is $X^{c*}_{/M_1}$. Moreover, $(g_f')^* \iota(X)$ is a constant of the motion of $\mathfrak{X}^{\omega_{M_f}}(M_f)$.

Proof. We shall first prove that X^c is projectable onto M_1, i.e., X^c satisfies

$$[X^c, \ker TLeg] \subset \ker TLeg$$

Since $\ker TLeg = \ker \omega_L \cap V(TQ)$, then, if $Z \in \ker \omega_L$, we have that

$$\begin{aligned} i_{[X^c, Z]}\omega_L &= L_{X^c} i_Z \omega_L - i_Z L_{X^c} \omega_L \\ &= -i_Z d i_{X^c} \omega_L - i_Z i_{X^c} d\omega_L \\ &= 0 \end{aligned}$$

Therefore, $[X^c, \ker \omega_L] \subset \ker \omega_L$. If we remember that $J[X^c, V] = 0$, for each vertical vector field on TQ, we deduce that X is projectable.

In local coordinates, we have

$$X^c = X^A \frac{\partial}{\partial q^A} + \dot{q}^B \frac{\partial X^A}{\partial q^B} \frac{\partial}{\partial \dot{q}^A}$$

Thus,

$$TLeg_1(X^c) = \left(X^A \frac{\partial}{\partial q^A} + \dot{q}^C \frac{\partial X^B}{\partial q^C} \frac{\partial^2 L}{\partial \dot{q}^B \partial \dot{q}^A} \frac{\partial}{\partial p_A} + X^B \frac{\partial^2 L}{\partial q^A \partial \dot{q}^B} \frac{\partial}{\partial p_A} \right)_{/M_1}$$

But, since $X^c L = 0$, we obtain

$$TLeg_1 X^c = \left(X^A \frac{\partial}{\partial q^A} - p_B \frac{\partial X^B}{\partial q^A} \frac{\partial}{\partial p_A} \right)_{/M_1} = X^{c*}_{/M_1}$$

Moreover,

$$i_{X^{c*}_{/M_1}} \omega_1 = d(g_1')^* \iota X$$

and

$$Leg_1^*(X^{c*}_{/M_1} H) = X^c(E_L)$$

Therefore, $X^{c*}_{/M_1} H = 0$. Now, from Corollary 4.1 and the Noether theorem, we obtain the result. ∎

Example 6.1. Consider the Lagrangian function $L: T\mathbb{R}^3 \to \mathbb{R}$ defined by

$$L = \frac{1}{2}(\dot{q}_1 + \dot{q}_2)^2$$

(see Krupková, 1994). Here (q^1, q^2, q^3) are the standard coordinates on $\mathbb{R}^3$ and $(q^1, q^2, q^3, \dot{q}^1, \dot{q}^2, \dot{q}^3)$ the induced ones on $T\mathbb{R}^3$.

The energy and the Poincaré–Cartan 1-form and 2-form are, respectively,

$$E_L = CL - L = \frac{1}{2}(\dot{q}_1 + \dot{q}_2)^2 = L$$

$$\alpha_L = (\dot{q}_1 + \dot{q}_2)\, dq_1 + (\dot{q}_1 + \dot{q}_2)\, dq_2$$

$$\omega_L = dq_1 \wedge d\dot{q}_1 + dq_1 \wedge d\dot{q}_2 + dq_2 \wedge d\dot{q}_1 + dq_2 \wedge d\dot{q}_2$$

It is easy to prove that ker ω_L is generated by

$$\frac{\partial}{\partial q_1} - \frac{\partial}{\partial q_2}, \quad \frac{\partial}{\partial q_3}, \quad \frac{\partial}{\partial \dot{q}_1} - \frac{\partial}{\partial \dot{q}_2}, \quad \frac{\partial}{\partial \dot{q}_3}$$

There are no secondary constraints, i.e., we have a global dynamics. The solutions of the equation

$$i_X \omega_L = dE_L$$

are given by

$$\mathfrak{X}^{\omega_L}(TQ) = \dot{q}_1 \frac{\partial}{\partial q_1} + \dot{q}_2 \frac{\partial}{\partial q_2} + \ker \omega_L$$

A function $F: TQ \to \mathbb{R}$ would be a constant of the motion of $\mathfrak{X}^{\omega_L}(TQ)$ if it satisfied the following equations:

$$\dot{q}_1 \frac{\partial F}{\partial q_1} + \dot{q}_2 \frac{\partial F}{\partial q_2} = 0, \qquad \frac{\partial F}{\partial q_1} - \frac{\partial F}{\partial q_2} = 0$$

$$\frac{\partial F}{\partial q_3} = 0, \qquad \frac{\partial F}{\partial \dot{q}_1} - \frac{\partial F}{\partial \dot{q}_2} = 0, \qquad \frac{\partial F}{\partial \dot{q}_3} = 0$$

Therefore, any function $F(\dot{q}_1, \dot{q}_2)$ such that $\partial F/\partial \dot{q}_1 = \partial F/\partial \dot{q}_2$ is a constant of the motion of $\mathfrak{X}^{\omega_L}(TQ)$. For instance,

$$F(q_1, q_2, q_3, \dot{q}_1, \dot{q}_2, \dot{q}_3) = \dot{q}_1 + \dot{q}_2$$

is a constant of the motion of $\mathfrak{X}^{\omega_L}(TQ)$. From the Noether theorem, we obtain the Cartan symmetries for the constant of the motion F, which are, precisely, the solutions of the equation

$$i_X\omega_L = dF$$

Then,

$$X = \frac{1}{2}\frac{\partial}{\partial q_1} + \frac{1}{2}\frac{\partial}{\partial q_2} + Z$$

where $Z \in \ker \omega_L$. In fact, $\partial/\partial q_1$ and $\partial/\partial q_2$ are infinitesimal symmetries of L. From Proposition 5.1 we have that

$$\alpha_L\left(\frac{\partial}{\partial q_1}\right) = \alpha_L\left(\frac{\partial}{\partial q_2}\right) = \dot{q}_1 + \dot{q}_2$$

is a constant of the motion, just F.

Now, we establish the Hamiltonian formulation for this example. Since

$$p_1 = \frac{\partial L}{\partial \dot{q}^1} = \dot{q}^1 + \dot{q}^2, \qquad p_2 = \frac{\partial L}{\partial \dot{q}^2} = \dot{q}^1 + \dot{q}^2, \qquad p_3 = \frac{\partial L}{\partial \dot{q}^1} = 0$$

we deduce that the submanifold M_1 of T^*Q is defined by the following primary constraints:

$$\phi_1 = p_1 - p_2 = 0 \qquad \text{and} \qquad \phi_2 = p_3 = 0$$

If we take coordinates (q^1, q^2, q^3, p_1) on M_1, we obtain that

$$\omega_{M_1} = (g_1')^*\omega_Q = dq^1 \wedge dp_1 + dq^2 \wedge dp_1$$

where

$$g_1'(q^1, q^2, q^3, p_1) = (q^1, q^2, q^3, p_1, p_1, 0)$$

The Hamiltonian energy H is

$$H = \frac{1}{2}p_1^2$$

Thus, $\ker \omega_{M_1}$ is generated by

$$\frac{\partial}{\partial q^3}, \quad \frac{\partial}{\partial q^1} - \frac{\partial}{\partial q^2}$$

and the solutions of the equation

$$i_X\omega_{M_1} = dH$$

are given by

$$\mathfrak{X}^{\omega_{M_1}}(M_1) = p_1 \frac{\partial}{\partial q^1} + \ker \omega_{M_1}$$

Since a function $F: M_1 \to \mathbb{R}$ is a constant of the motion of $\mathfrak{X}^{\omega_{M_1}}(M_1)$ if and only if

$$p_1 \frac{\partial F}{\partial q^1} = 0, \qquad \frac{\partial F}{\partial q^3} = 0, \qquad \frac{\partial F}{\partial q^1} - \frac{\partial F}{\partial q^2} = 0$$

we deduce that F has to be of the form $F = F(p_1)$.

7. THE SECOND-ORDER DIFFERENTIAL EQUATION PROBLEM

Let Z be a vector field on M_f such that

$$(i_Z\omega_1 = dh_1)_{/M_f}$$

We know that

$$P_f/\ker TLeg_f \cong M_f$$

Given a vector field X on P_f which projects onto Z, we can find a unique point y in each fiber of Leg_f such that X satisfies at y the SODE condition, i.e., $(JX)_y = C_y$.

In local coordinates, if X is locally written as

$$X = X^A \frac{\partial}{\partial q^A} + \tilde{X}^A \frac{\partial}{\partial \dot{q}^A}$$

since $z = Leg_f(y) \in M_f$, and we identify z with the fiber which contains y, we deduce that X^A is constant on the fiber. Moreover,

$$U = JX - C = (X^A - \dot{q}^A) \frac{\partial}{\partial \dot{q}^A}$$

is tangent to the fibers. Let $\sigma(t) = (q^A(t), \dot{q}^A(t))$ be the integral curve of U which contains the point y with coordinates $(q_0^A, \dot{q}_0^A)$. We deduce that

$$\sigma(t) = (q_0^A, X^A - e^{-t}(X^A - \dot{q}_0^A))$$

We then obtain

$$\bar{y} = \lim_{t \to \infty} \sigma(t) = (q_0^A, X^A)$$

Thus, the point $\bar{y}$ with coordinates (q_0^A, X^A) is in the same fiber as y, since the fibers are closed. Moreover, $U(\bar{y}) = 0$, and, therefore X satisfies the SODE condition at the point $\bar{y}$.

We obtain a differentiable section α: $M_f \to P_f$ of Leg_f and its image $S = \alpha(M_f)$ is a submanifold of P_f, on which X satisfies the SODE condition. In general, X is not tangent to S, but the vector field $\xi' = T\alpha(Z)$ is tangent to S, it is a solution of the equation

$$(i_X\omega_L = dE_L)_{/S}$$

and it also satisfies the SODE condition.

Now we study the relationship between the symmetries and constant of the motion defined on S and the ones defined on P_f.

Proposition 7.1. If $Z_{\omega_{M_f}}$ is a solution of the equation

$$i_X\omega_{M_f} = dH_{/M_f} \tag{9}$$

then the vector field $\xi' = T\alpha(Z_{\omega_{M_f}})$ on S is a solution of the equation

$$i_X\omega_S = dE_{L/S} \tag{10}$$

where $\omega_S = j^*\omega_L$ and j is the embedding of S into TQ.

Conversely, if ξ' is a solution of (9), then $Z = T\alpha^{-1}(\xi')$ is a solution of (10).

Proof. In fact,

$$\begin{aligned} i_{\xi'}\omega_S &= i_{T\alpha Z}(\alpha^*\omega_{M_f}) \\ &= (\alpha^{-1})^*(i_Z\omega_{M_f}) \end{aligned}$$

and, since

$$dE_{L/S} = (\alpha^{-1})^*(dH_{/M_f})$$

we obtain the required result. ■

Since S and M_f are diffeomorphic and the dynamics on them are equivalent, there exists a complete equivalence between symmetries and constants of the motion via α as well as via Leg_f/S: $S \to M_f$.

8. GENERALIZED HAMILTONIAN DYNAMICS

In this section, we study the relationship between the symmetries and constants of the motion for a regular Lagrangian system on TQ and the symmetries and constants of the motion for the presymplectic system defined on $T^*Q \oplus TQ$. This formulation using the space $T^*Q \oplus TQ$ was established by Skinner and Rusk (1983a,b) (see also Cariñena *et al.*, 1988; López, 1989; de León and Rodrigues, 1989).

Let Q be an n-dimensional differentiable manifold. Consider the Whitney sum of T^*Q with TQ, denoted by

$$W_0 = T^*Q \oplus TQ$$

Let

$$\pi_1: T^*Q \oplus TQ \to T^*Q$$
$$\pi_2: T^*Q \oplus TQ \to TQ$$

be the projections onto the first and the second factors, respectively.

Let $L: TQ \to \mathbb{R}$ be a regular Lagrangian with energy E_L. The Poincaré–Cartan 2-form ω_L is therefore symplectic and the Legendre transformation is a local diffeomorphism. If we suppose, moreover, that the Lagrangian L is hyperregular, then the Legendre transformation Leg: $TQ \to T^*Q$ is a diffeomorphism. We denote by ξ_L the Euler–Lagrange vector field, by ω_Q the canonical symplectic form on T^*Q, and by X_H the Hamiltonian vector field with energy H. We have

$$(Leg^{-1})^*E_L = H$$
$$TLeg(\xi_L) = X_H$$
$$(Leg^{-1})^*\omega_L = \omega_Q$$

Define on $W_0 = T^*Q \oplus TQ$ a presymplectic 2-form $\omega = \pi_1^*\omega_Q$ and a function $D: W_0 \to \mathbb{R}$ by

$$D = \langle \pi_1, \pi_2 \rangle - \pi_2^* L$$

If (q^A) are local coordinates on a neighborhood U of Q, $(q^A, \dot{q}^A)$ the induced coordinates on TU, and (q^A, p_A) the induced coordinates on T^*U, then we denote by $(q^A, p_A, \dot{q}^A)$ the induced coordinates on $T^*U \oplus TU$. Locally, D is written as follows:

$$D(q^A, p_A, \dot{q}^A) = p_A \dot{q}^A - L(q^A, \dot{q}^A)$$

We obtain a presymplectic system (W_0, ω, dD) and we apply the constraint algorithm to it (López, 1989; Cariñena *et al.*, 1988). The constraint submanifold W_1 of W_0 is just $W_1 = \text{Graph}(Leg)$, and we denote by j_1 the embedding

$$W_1 \xrightarrow{j_1} W_0$$

W_1 is locally characterized by the constraints

$$\phi_A = p_A - \frac{\partial L}{\partial \dot{q}^A}, \qquad A = 1, \ldots, n$$

If we consider the 2-form $j_1^*\omega = \omega_{W_1}$ on W_1, we obtain that

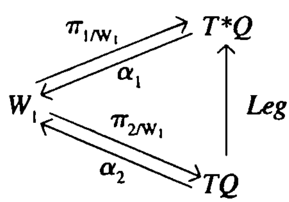
$$\ker \omega_{W_1} = TW_1^{\perp} \cap TW_1$$

Since $TW_1^{\perp}$ is locally generated by $\{\partial/\partial\dot{q}^A\}$, $A = 1, \ldots, n$, then a vector field X is tangent to W_1 if and only if $X(\phi_A)_{/W_1} = 0$, for any A. Since L is regular, $\partial/\partial\dot{q}_A$, $A = 1, \ldots, n$, is not tangent to W_1 and thus $\ker \omega_{W_1} = 0$. Hence, (W_1, ω_{W_1}) is a symplectic manifold.

Since π_{1/W_1} and π_{2/W_1} are diffeomorphisms, we can consider the inverse maps α_1 and α_2 of these projections. They are defined as follows:

$$\alpha_1: \qquad T^*Q \to W_1$$
$$(q^A, \dot{q}^A) \mapsto \left(q^A, \frac{\partial H}{\partial p_A}, p_A\right)$$
$$\alpha_2: \qquad TQ \to W_1$$
$$(q^A, \dot{q}^A) \mapsto \left(q^A, \dot{q}^A, \frac{\partial L}{\partial \dot{q}^A}\right)$$

We obtain the following commutative diagram:



The condition of a vector field X to be tangent to W_1 may be written as follows. If

$$X = X^A \frac{\partial}{\partial q^A} + X'_A \frac{\partial}{\partial p_A} + X''^A \frac{\partial}{\partial \dot{q}^A}$$

then X is tangent to W_1 if and only if

$$X^A \frac{\partial^2 L}{\partial q^A \partial \dot{q}^B} + X''^A \frac{\partial^2 L}{\partial \dot{q}^A \partial \dot{q}^B} = X'_k, \qquad k = 1, \ldots, n$$

Since α_2 is a diffeomorphism, the vector field $X = T\alpha_2(\xi_L)$ is well defined. Locally, if

$$\xi_L = \dot{q}^A \frac{\partial}{\partial q^A} + \xi^A \frac{\partial}{\partial \dot{q}^A}$$

we obtain

$$X = \left(\dot{q}^A \frac{\partial}{\partial q^A} + \xi^A \frac{\partial}{\partial \dot{q}^A} + \left(\dot{q}^B \frac{\partial^2 L}{\partial \dot{q}^B \partial q^A} + \xi^B \frac{\partial^2 L}{\partial \dot{q}^B \partial \dot{q}^A} \right) \frac{\partial}{\partial p_A} \right)_{/W_1}$$

Moreover, X is the solution of the equation

$$i_X \omega_{W_1} = dj_1^* D$$

i.e., X is the Hamiltonian vector field $X_{j_1^* D}$.

We will now study the relationship between the symmetries and constants of the motion of these systems.

Proposition 8.1. A function g: $TM \to \mathrm{R}$ is a constant of the motion of ξ_L if and only if $(\pi_{2/W_1})^* g$ is a constant of the motion of $X_{j_1^* D}$.

Proof. In fact, since $X_{j_1^* D} = (\alpha_2)_*(\xi_L)$, we have

$$X_{j_1^* D}((\pi_{2/W_1})^* g) = (\pi_{2/W_1})^*(\xi_L g) = 0$$

The converse is trivial, because π_{2/W_1} is a diffeomorphism. ■

Corollary 8.1. A function g: $T^*M \to R$ is a constant of the motion of X_H if and only if $(\pi_{1/W_1})^* g$ is a constant of the motion of $X_{j_1^* D}$.

Proposition 8.2. A vector field $\tilde{X}$ is a dynamical symmetry of $X_{j_1^* D}$ if and only if $(\pi_{1/W_1})_* \tilde{X}$ and $(\pi_{2/W_1})_* \tilde{X}$ are dynamical symmetries of X_H and of ξ_L, respectively.

Proof. In fact, we have the following equivalences:

$$[\tilde{X}, X_{j_1^* D}] = 0 \Leftrightarrow [(\pi_{1/W_1})_* \tilde{X}, X_H] = 0 \Leftrightarrow [(\pi_{2/W_1})_* \tilde{X}, \xi_L] = 0 \quad ■$$

Proposition 8.3. A vector field $\tilde{X}$ on W_1 is a Cartan symmetry for the presymplectic system $(W_1, \omega_{W_1}, dj_1^* D)$ if and only if $(\pi_{1/W_1})_* \tilde{X}$ is a Cartan symmetry of (T^*Q, ω_Q, dH) or, equivalently, if $(\pi_{2/W_1})_* \tilde{X}$ is a Cartan symmetry of (TQ, ω_L, dE_L).

Proof. In fact, if

$$i_{\tilde{X}} \omega_{W_1} = dF \qquad \text{with} \quad F\colon W_1 \to \mathrm{R}$$

then

$$\alpha_1^*(i_{\tilde{X}} \omega_{W_1}) = i_{(\pi_{1/W_1})_* \tilde{X}} \omega_Q = d(\alpha_1^* F)$$

In a similar way, by applying α_2 we obtain that

$$i_{(\pi_{2/W_1})_* \tilde{X}}\omega_L = d(\alpha_2^* F)$$

Moreover,

$$\tilde{X}(j_1^* D) = 0 \Leftrightarrow ((\pi_{1/W_1})_* \tilde{X})H = 0 \Leftrightarrow ((\pi_{2/W_1})_* \tilde{X})E_L = 0 \quad \blacksquare$$

Let X be a vector field on Q. There exists a unique vector field $X^{(c,c^*)}$ on W_0 such that

$$(\pi_1)_* X^{(c,c^*)} = X^{c^*} \qquad \text{and} \qquad (\pi_2)_* X^{(c,c^*)} = X^c$$

In local coordinates, if

$$X^c = X^A \frac{\partial}{\partial q^A} + \dot{q}^B \frac{\partial X^A}{\partial q^B} \frac{\partial}{\partial \dot{q}^A}$$

$$X^{c^*} = X^A \frac{\partial}{\partial q^A} - p_B \frac{\partial X^B}{\partial q^A} \frac{\partial}{\partial p_A}$$

then

$$X^{(c,c^*)} = X^A \frac{\partial}{\partial q^A} - p_B \frac{\partial X^B}{\partial q^A} \frac{\partial}{\partial p_A} + \dot{q}^B \frac{\partial X^A}{\partial q^B} \frac{\partial}{\partial \dot{q}^A}$$

Thus, $X^{(c,c^*)}$ is tangent to W_1 if and only if

$$\left(X^A \frac{\partial^2 L}{\partial q^A \partial \dot{q}^B} + \dot{q}^A \frac{\partial X^A}{\partial q^B} \frac{\partial^2 L}{\partial \dot{q}^A \partial \dot{q}^B} \right)_{/W_1} = \left(-p_B \frac{\partial X^B}{\partial q^A} \right)_{/W_1} \tag{11}$$

i.e., if and only if X^c and X^{c^*} are *Leg*-related. Therefore, if X is an infinitesimal symmetry of L, it satisfies (11).

Proposition 8.4. Let X be a vector field on Q. Then, we have

$$X^c L = 0 \Leftrightarrow X^{c^*} H = 0 \Leftrightarrow (X^{(c,c^*)} D)_{/W_1} = 0$$

9. AFFINE LAGRANGIANS

Let Q be an n-dimensional differentiable manifold. Consider a function h: $Q \to \mathbb{R}$ and a 1-form μ on Q. We obtain an affine Lagrangian function on TQ as follows:

$$L = \hat{\mu} + h^v$$

where $\hat{\mu}$: $TQ \to \mathbb{R}$ is the function defined by $\hat{\mu}(x, u) = \langle \mu(x), u \rangle$ and $h^v(x, u) = h(x)$ with $u \in T_x Q$. If $\mu = a_A(q)\, dq^A$, we obtain

$$L = a_A(q) \dot{q}^A + h$$

Thus,

$$E_L = -h^v$$

$$\alpha_L = -\mu^v$$

$$\omega_L = d\mu^v$$

Since $V(TQ) \subset \ker \omega_L$, we have

$$\dim \ker \omega_L \leq 2 \dim(V_p \ker \omega_L)$$

Therefore, L is a Lagrangian of type III according to the classification by Cantrijn *et al.* (1986).

We apply the constraint algorithm to the presymplectic system (TQ, ω_L, dE_L).

For the sake of simplicity, we shall assume that the 2-form $d\mu$ is symplectic. Of course, we may analyze the general case when $d\mu$ is degenerate, but nothing especially new will be obtained. Then, there exists a unique vector field X_h such that

$$i_{X_h} d\mu = dh$$

X_h is the Hamiltonian vector field with energy h. Thus, the presymplectic system (TQ, ω_L, dE_L) has a global dynamics, since the complete lift X_h^c of X_h is a solution of the equation

$$i_X(-\omega_L) = dE_L \tag{12}$$

The set of solutions of (12) is

$$\mathfrak{X}^{\omega_L}(TQ) = X_h^c + V(TQ)$$

Since the presymplectic system (TQ, ω_L, E_L) has a global dynamics, all the definitions and results obtained in Section 2 are applicable to it.

The Legendre transformation is defined by

$$\begin{aligned} Leg\colon\ & TQ \to T^*Q \\ & (q^A, \dot{q}^A) \mapsto (q^A, a_A) \end{aligned}$$

The map

$$\begin{aligned} \phi\colon\ & Q \to Leg(TQ) = M_1 \\ & (q^A) \mapsto (q^A, a_A) \end{aligned}$$

is a diffeomorphism and $\operatorname{Im} \mu = M_1$, $Leg = \tau_Q \circ \phi$. We deduce that Leg is a submersion with connected fibers. Therefore, L is almost regular. If we denote by $j_1\colon M_1 \to T^*Q$ the embedding, since ϕ is a diffeomorphism, we obtain that the 2-form $j_1^*\omega_Q = \omega_{M_1}$ is symplectic. If we define $H: M_1 \to \mathbb{R}$ by $H \circ Leg = E_L$, we have that $\phi^*H = -h$ and, moreover, $\phi^*\omega_{M_1} = d\mu$. Then, the Hamiltonian vector fields X_H and $-X_h$ are ϕ-related because the symplectic structures $d\mu$ and $(\omega_{M_s}$ are symplectomorphic. Thus, the study of the symmetries and constants of the motion for both systems is equivalent.

Our purpose is to find a relationship between the symmetries and constants of the motion for the symplectic system $(Q, d\mu, dh)$ and the presymplectic system (TQ, ω_L, dE_L).

Proposition 9.1. If F is a constant of the motion of X_h, then F^v is a constant of the motion of $\mathfrak{X}^{\omega_L}(TQ)$. Conversely, if G is a constant of the motion of $\mathfrak{X}^{\omega_L}(TQ)$, then G is projectable onto Q and its projection is a constant of the motion of X_h.

Proof. In fact, if $X_hF = 0$, then $V(TQ)F^v = 0$ and also $X_h^cF^v = 0$. Thus, F^v is a constant of the motion of $\mathfrak{X}^{\omega_L}(TQ)$. Conversely, if G is a constant of the motion of $\mathfrak{X}^{\omega_L}(TQ)$, since $V(TQ)G = 0$, we have that G is projectable. If we denote by g the projection of G onto Q, then, since $X_h^cG = 0$, we deduce that $X_hg = 0$. ■

Proposition 9.2. Let X be a vector field on Q. If X is a dynamical symmetry of X_h, then X^c is a dynamical symmetry of $\mathfrak{X}^{\omega_L}(TQ)$.

Proof. If $[X, X_h] = 0$, then $[X^c, X_h^c] = 0$. We also have that $[X^c, V] \in V(TQ)$, $\forall V \in V(TQ)$. Hence, we deduce that

$$[X^c, \mathfrak{X}^{\omega_L}(TQ)] \subset V(TQ) = \ker \omega_L \quad ■$$

Proposition 9.3. If X is a Cartan symmetry of the presymplectic system $(Q, d\mu, dh)$, then X^c is a Cartan symmetry of (TQ, ω_L, dE_L), and conversely.

Proof. In fact, if X is a Cartan symmetry of $(Q, d\mu, dh)$, we have

$$i_X\, d\mu = df \qquad \text{and} \qquad Xh = 0$$

where f is a function on Q. But this holds if and only if

$$i_{X^c}\, d\mu^v = df^v \qquad \text{and} \qquad X^ch^v = 0$$

i.e., if X^c is a Cartan symmetry of the presymplectic system $(TQ, \omega_L = d\mu^v, dE_L = -dh^v)$. ■

From Proposition 9.3 we obtain, as a corollary, Theorem 2 of Cariñena *et al.* (1988). This theorem states that if L is a regular Lagrangian and ξ_L is the Euler–Lagrange vector field, there exists a one-to-one correspondence between the constants of the motion of ξ_L and the Noether symmetries of the presymplectic system $(TTQ, d\alpha_L^v, dE_L^v)$.

10. REDUCTION OF A DEGENERATE LAGRANGIAN SYSTEM

Let L: $TQ \rightarrow R$ be a degenerate Lagrangian. We suppose that it satisfies the following assumptions (Cantrijn *et al.*, 1986):

1. The Poincaré–Cartan 2-form ω_L is presymplectic, that is, it has constant rank.
2. The Lagrangian L admits a global dynamics.
3. The foliation defined by ker ω_L is a fibration. In such a case, the leaf space $TQ/\ker \omega_L = (TQ)_0$ admits a manifold structure.

From these assumptions, there exists a unique symplectic form $\tilde{\omega}_L$ on the manifold $(TQ)_0$ such that $\omega_L = (\pi_L)^*(\tilde{\omega}_L)$, where π_L is the projection π_L: $TQ \rightarrow (TQ)_0$. Moreover, the energy function E_L is also projectable because (ker $\omega_L)E_L = 0$. We denote by $\tilde{E}_L$ its projection. Since $((TQ)_0, \tilde{\omega}_L)$ is a symplectic manifold, then there exists a unique vector $\tilde{\xi}$ such that it satisfies the equation

$$i_{\tilde{\xi}}\tilde{\omega}_L = d\tilde{E}_L$$

and each vector field $\xi \in \mathfrak{X}^{\omega_L}(TQ)$ projects onto $\tilde{\xi}$, i.e., $T\pi_L(\xi) = \tilde{\xi}$.

The following results, which give the relationship between the constants of the motion and symmetries for the presymplectic system (TQ, ω_L, dE_L) and the symplectic system $((TQ)_0, \tilde{\omega}_L, d\tilde{E}_L)$, can be proved by a direct computation.

Proposition 10.1. If f: $TQ \rightarrow R$ is a constant of the motion of $\mathfrak{X}^{\omega_L}(TQ)$, then f is projectable onto a function $\tilde{f}$ which is also a constant of the motion of $\tilde{\xi}$. Conversely, if $\tilde{f}$: $(TM)_0 \rightarrow R$ is a constant of the motion of $\tilde{\xi}$, then $(\pi_L)^*\tilde{f}$ is a constant of the motion of $\mathfrak{X}^{\omega_L}(TQ)$.

Proposition 10.2. If X is a dynamical symmetry of $\mathfrak{X}^{\omega_L}(TQ)$, then X is projectable and its projection $\tilde{X}$ is a dynamical symmetry of $\tilde{\xi}$. Conversely, if $\tilde{X}$ is a dynamical symmetry of $\tilde{\xi}$, then any vector field on TQ that is projectable onto $\tilde{X}$ is a dynamical symmetry of $\mathfrak{X}^{\omega_L}(TQ)$.

Proposition 10.3. If X is a Cartan symmetry of (TQ, ω_L, dE_L), then X is projectable and its projection $\tilde{X}$ is a Cartan symmetry of $((TQ)_0, \tilde{\omega}_L, d\tilde{E}_L)$. Conversely, if $\tilde{X}$ is a Cartan symmetry of $((TQ)_0, \tilde{\omega}_L, d\tilde{E}_L)$, then any vector field on TQ that is projectable onto $\tilde{X}$ is a Cartan symmetry of (TQ, ω_L, dE_L).

If we suppose, moreover, that ker ω_L is a tangent distribution [i.e., ker ω_L is the natural lift of a distribution D on Q (Cantrijn *et al.*, 1986)], then the canonical almost tangent structure J and the Liouville vector field C on TQ project onto an integrable almost tangent structure J_0 on $(TQ)_0$ and onto a vector field C_0 such that

$$J_0C_0 = 0, \qquad L_{C_0}J_0 = -J_0$$

respectively. De León *et al.* (1994) proved that $(TQ)_0$ has the unique structure of a vector bundle which is isomorphic to the tangent bundle TS of the singular manifold S of C_0 and this isomorphism transports the canonical almost tangent structure and the Liouville vector field of TS to J_0 and C_0, respectively (see also de Filippo *et al.*, 1989). We denote by ϕ the isomorphism

$$\phi\colon TS \to (TM)_0$$

Proposition 10.4. Let X be a vector field on Q. If X^c is π_L-projectable, then there exists a vector field Y on S such that

$$T\phi(Y^c) = T\pi_L(X^c)$$

Proof. Since ker ω_L is an involutive tangent distribution, we have ker $\omega_L = D^c$, where D is an involutive distribution on Q. From the Frobenius theorem, there exist local coordinates (x^A, x^B), $1 \leq A \leq k$ and $k + 1 \leq B \leq n$, around each point of Q such that

$$D = \left\langle \frac{\partial}{\partial x^B} \right\rangle$$

Therefore

$$\ker \omega_L = \left\langle \frac{\partial}{\partial x^B}, \frac{\partial}{\partial \dot{x}^B} \right\rangle$$

Thus, in these coordinates, we can write the projection π_L as

$$\pi_L(x^A, x^B, \dot{x}^A, \dot{x}^B) = (x^A, \dot{x}^A)$$

By the definition of S, we can find local coordinates $(y^A, \dot{y}^A)$ such that the isomorphism ϕ is locally expressed by

$$\phi(y^A, \dot{y}^A) = (x^A, \dot{x}^A)$$

By using these local coordinates, the result follows from a direct computation. ∎

Moreover, if Y is a vector field on S, there exists a vector field X on Q such that

$$T\pi_L(X^c) = T\phi(Y^c)$$

In fact, for each vector field X' on Q, with $X' = X + Z$, where Z belongs to D, we have

$$T\pi_L((X')^c) = T\phi(Y^c)$$

We denote by $\omega_S = \phi^*\tilde{\omega}_L$ and by $g = \phi^*\tilde{E}_L$ the pullbacks of $\tilde{\omega}_L$ and

$\tilde{E}_L$, respectively. Therefore (TS, ω_S, dg) is a Hamiltonian system, and there exists a unique vector field ξ_S such that

$$i_{\xi_S}\omega_S = dg$$

Obviously, we have that $\xi_S = (T\phi)^{-1}(\tilde{\xi})$. From Proposition 10.4, we deduce the following result.

Proposition 10.5. Let X be a vector field on Q.

1. If X is a Lie symmetry of $\mathfrak{X}^{\omega_L}(TQ)$, then there exists a vector field Y on S such that Y^c is a dynamical symmetry of ξ_S.
2. If X is a Noether symmetry of L, then there exists a vector field Y on S such that Y^c is a Cartan symmetry of the symplectic system (TS, ω_S, dg).

Example 10.1. We shall apply the previous results to the electron-monopole system (Marmo, 1988; de Filippo *et al.*, 1989; de León *et al.*, 1994). The equations of motion are

$$\begin{cases} \dfrac{dx^i}{dt} = v^i \\ \dfrac{dv^i}{dt} = \dfrac{n}{r^3}\, \epsilon^i_{jk} x^j v^k \end{cases} \tag{13}$$

where r is the distance of the point to the origin $0 \in \mathbb{R}^3$, $n = eg/4\pi m$ is the product of the electric and magnetic charges divided by 4π times the mass of the electron, and

$$\epsilon^i_{jk} = \begin{cases} 1 & ijk \text{ in cyclic order} \\ -1 & ijk \text{ in anticyclic order} \\ 0 & \text{if two indices are equal} \end{cases}$$

This system does not admit a global Lagrangian description. We can solve this problem by enlarging the configuration space to $\mathbb{R} \times SU(2)$. Then we can define a global Lagrangian L such that its motion equations project onto equations (13). In fact, consider the Hopf fibration:

$$\pi_{\mathrm{H}}: SU(2) \to S^2$$

and the induced projection

$$T(\mathrm{id}_{\mathbb{R}} \times \pi_{\mathrm{H}}): T(\mathbb{R} \times SU(2)) \to T(\mathbb{R} \times S^2)$$

Define a global Lagrangian on $T(\mathrm{R} \times SU(2))$ by

$$L(r, s) = \frac{1}{2}\left(\dot{r}^2 + r^2 \sum_{j=1}^{3} (\dot{x}^j)^2\right) + ni \operatorname{tr} \sigma^3 s^{-1}\dot{s}$$

Here, r denotes the coordinate in R and s an element of $SU(2)$, x^i is the i-coordinate of $\pi_H(s)$, and σ^3 is the usual Pauli matrix.

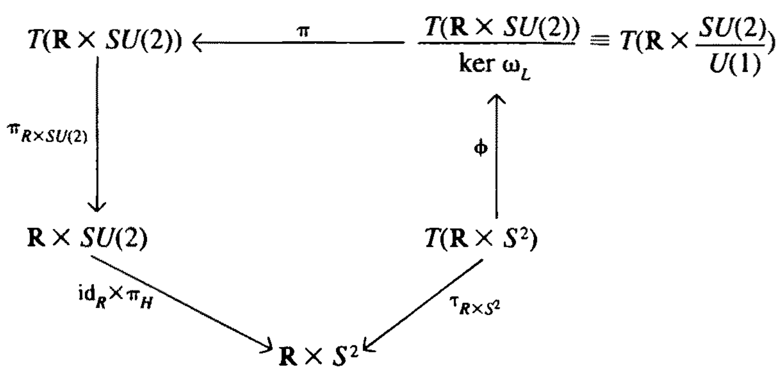
Now, consider the presymplectic system

$$(T(\mathrm{R} \times SU(2)), \omega_L, dE_L)$$

with ω_L the Poincaré–Cartan 2-form and E_L the energy function. $\ker \omega_L$ is generated by

$$\{X_3^{v}, X_3^{c}\}$$

where X_3 is the fundamental vector field of $U(1) \equiv S^1$ in the Hopf bundle π_H: $SU(2) \to S^2$. Moreover, we have that the presymplectic system admits a global dynamics and the foliation defined by $\ker \omega_L$ is a fibration. Then, the quotient manifold admits a unique vector bundle structure such that it is isomorphic to a tangent bundle. Thus, we obtain the following diagram:



Electron–monopole system

From Propositions 10.4 and 10.5, we can relate the symmetries and constants of the motion of the presymplectic system

$$(T(\mathrm{R} \times SU(2)), \omega_L, dE_L)$$

with the corresponding ones of the symplectic system

$$(T(\mathrm{R} \times S^2), (\phi^{-1})^*\tilde{\omega}_L, (\phi^{-1})^*\tilde{E}_L)$$

where $\tilde{\omega}_L$ and $\tilde{E}_L$ are the projections of ω_L and E_L, respectively.

If we consider coordinates (y^0, y^1, y^2, y^3) on $SU(2)$, then a generic element s of $SU(2)$ can be represented by

$$s = y^0 \mathrm{I} + \sum_{k=1}^{3} i y^k \sigma^k$$

where σ^1, σ^2, and σ^3 are Pauli matrixes and these coordinates satisfy the constraint $(y^0)^2 + (y^1)^2 + (y^2)^2 + (y^3)^2 = 1$. Now, let $\{Y^1, Y^2, Y^3\}$ be the basis of right-invariant vector fields of the Lie group $SU(2)$:

$$Y^i = y^0 \frac{\partial}{\partial y^i} - y^i \frac{\partial}{\partial y^0} - \sum_{k=1}^{3} \epsilon^i_{jk} y^j \frac{\partial}{\partial y^k}, \qquad 1 \leq i \leq 3$$

If we denote by the same letters the induced vector fields on $\mathbb{R} \times SU(2)$, then it is easy to prove that

$$(Y^i)^c L = 0$$

that is, (Y^i) is an infinitesimal symmetry of L. From Proposition 5.1, we have that $(Y^i)^v L$ is a constant of the motion of $\mathfrak{X}^{\omega_L}(T(\mathbb{R} \times SU(2)))$. Also, from Proposition 10.5, the vector field $\phi_*(\tilde{Y}^i)$ is a Cartan symmetry of the symplectic system $(T(\mathbb{R} \times S^2), (\phi^{-1})^*\tilde{\omega}_L, (\phi^{-1})^*\tilde{E}_L)$, and the constants of the motion $(Y^i)^v L$ project onto three constants of the motion f^i, $1 \leq i \leq 3$, of ξ_S, where ξ_S is the Hamiltonian vector field of $(\phi^{-1})^*\tilde{E}_L$. In fact, we have that

$$(Y^i)^v L = \epsilon^i_{jk} x^j \dot{x}^k + \frac{n x^i}{r}$$

which shows that it is projectable and its projection is a constant of the motion of the projected symplectic system. Since f^i is a constant of the motion of ξ_S, from the Noether theorem, there exists a vector field X^i (the Hamiltonian vector field of f^i) which is a Cartan symmetry of the symplectic system. A straightforward computation shows that these vector fields X^i are precisely the complete lifts of the infinitesimal generators

$$X^i = \sum_{k=1}^{3} \epsilon^i_{jk} x^j \frac{\partial}{\partial x^k}, \qquad 1 \leq i \leq 3$$

obtained from the canonical basis of $\mathfrak{so}(3)$, the Lie algebra of $SO(3)$, that is, the rotations in $\mathbb{R}^3$.

11. DEGENERATE LAGRANGIAN SYSTEMS WITH A LIE GROUP OF SYMMETRIES

Let (M, ω) be a presymplectic manifold, i.e., ω is a closed 2-form with constant rank. We suppose that we have a presymplectic action of a Lie group G on a manifold M:

$$\phi: G \times M \to M$$

i.e., $\phi_g^*\omega = \omega$, $\forall g \in G$ (Binz *et al.*, 1988). If we denote by $\mathfrak{g}$ the Lie algebra of the Lie group G, a *momentum map* is a map $J: M \to \mathfrak{g}^*$ such that for all $\xi \in \mathfrak{g}$ the vector field ξ_M (the infinitesimal generator of the flow $\phi_{\exp t\xi}$) is a Hamiltonian vector field of $J\xi = \langle J, \xi \rangle$, i.e.,

$$i_{\xi_M}\omega = d(J\xi)$$

Proposition 11.1. Let (M, ω, dF) be a presymplectic system. We consider a presymplectic action $\phi: G \times M \to M$ of a Lie group G which admits a momentum map and such that F is G-invariant. If the constraint algorithm for the presymplectic system (M, ω, dF) ends in a final constraint submanifold M_f and $j: M_f \to M$ is the embedding of M_f into M, then we have that, for all $\xi \in \mathfrak{g}$, the map $j^*(J\xi): M_f \to \mathbb{R}$ is a constant of the motion of $\mathfrak{X}^{\omega_{M_f}}(M_f)$ and therefore it is a constant of the motion of $\mathfrak{X}^{\omega}(M_f)$.

Proof. Let $J: M \to \mathfrak{g}^*$ be a momentum map. Hence

$$i_{\xi_M}\omega = d(J\xi)$$

and, moreover, from $\phi_g^* F = F$, $\forall g \in G$, we have

$$L_{\xi_M}F = \xi_M F = 0$$

Now we apply Corollary 4.1. ∎

We now consider the action of a Lie group G on a manifold Q:

$$\phi: G \times Q \to Q$$

and lift this action to an action on TQ:

$$\phi^T: G \times TQ \to TQ$$

as follows:

$$(\phi^T)_g: TQ \to TQ, \quad (\phi^T)_g = T\phi_g$$

We suppose that $L: TQ \to \mathbb{R}$ is a G-invariant Lagrangian, i.e., $L \circ \phi_g^T = L$, $\forall g \in G$. We have that E_L, α_L, and ω_L are invariants by ϕ^T and we deduce that $\forall \xi \in \mathfrak{g}$ the vector field ξ_Q is an infinitesimal symmetry of L. From Proposition 5.1, we deduce that $((\xi_Q)^v L)_{/P_f}$ is a constant of the motion of $\mathfrak{X}^{\omega_{P_f}}(P_f)$ (here, P_f is the final constraint submanifold for the presymplectic system (TQ, ω_L, dE_L)).

12. SYMMETRIES AND CONSTANTS OF THE MOTION FOR PRECOSYMPLECTIC SYSTEMS

Let (M, Ω, η) be a precosymplectic system, i.e., Ω is a closed 2-form with constant rank $2r$ and η a closed 1-form (Chinea *et al.*, 1994).

The dynamic for this precosymplectic system (M, Ω, η) is given by introducing a Hamiltonian function $H: M \to \mathbb{R}$. We consider the modified precosymplectic system $(M, \Omega_H = \Omega + dH \wedge \eta, \eta)$. The 2-form Ω_H is obviously closed, although it may not have constant rank. In fact, all that we obtain is that

$$2r \leq \text{rank } \Omega_H \leq 2r + 2$$

We distinguish two cases:

1. First Case. Ω_H has constant rank $2r$, that is, (Ω_H, dt) is a precosymplectic structure.

Then, the equations

$$i_X\Omega_H = 0, \qquad i_X\eta = 1 \tag{14}$$

have globally defined solutions, i.e., there exists a vector field ξ on M such that satisfies (14). It is clear that any vector field

$$\xi + \ker \Omega_H \cap \ker \eta$$

is a solution of (14). We denote by $\mathfrak{X}^{(\Omega_H,\eta)}(M)$ the set of all the solutions of (14), i.e.,

$$\mathfrak{X}^{(\Omega_H,\eta)}(M) = \{\xi + Z/Z \in \ker \Omega_H \cap \ker \eta\}$$

Definition 12.1. A constant of the motion of $\mathfrak{X}^{(\Omega_H,\eta)}(M)$ is a function F: $M \to \mathbb{R}$ such that $\mathfrak{X}^{(\Omega_H,\eta)}(M)F = 0$.

Remark 12.1. A constant of the motion of $\mathfrak{X}^{(\Omega_H,\eta)}(M)$ satisfies that $(\ker \Omega_L \cap \eta)F = 0$.

The following lemma allows us to give a useful characterization of the constants of the motion of $\mathfrak{X}^{(\Omega_H,\eta)}(M)$.

Lemma 12.1. If a function $F: M \to \mathbb{R}$ is a constant of the motion of $\mathfrak{X}^{(\Omega_H,\eta)}(M)$ then we have that

$$(\ker \Omega_H)F = 0$$

Proof. In fact, if $Z \in \ker \Omega_H$, we obtain

$$Z = (Z - \eta(Z)\xi) + \eta(Z)\xi$$

for any solution ξ of (14). Since $Z - \eta(Z)\xi \in \ker \Omega_H \cap \ker \eta$, from Remark 12.1, we deduce that, if F is a constant of the motion of $\mathfrak{X}^{(\Omega_H,\eta)}(M)$, then

$$ZF = (Z - \eta(Z)\xi)F + \eta(Z)(\xi F) = 0 \quad \blacksquare$$

Therefore, we can characterize the constants of the motion as the functions F such that they satisfy the property

$$(\ker \Omega_H)F = 0$$

Definition 12.2. A vector field X on M is said to be a dynamical symmetry of $\mathfrak{X}^{(\Omega_H,\eta)}(M)$ if

$$[X, \ker \Omega_H] \subset \ker \Omega_H$$

Remark 12.2. If X is a dynamical symmetry of $\mathfrak{X}^{(\Omega_H,\eta)}(M)$, then any vector field Y such that $Y = X + Z$, with $Z \in \ker \Omega_H$, is also a dynamical symmetry of $\mathfrak{X}^{(\Omega_H,\eta)}(M)$. In fact, for any $Z' \in \ker \Omega_H$ we have

$$\begin{aligned} i_{[Z,Z']}\Omega_H &= L_Z i_{Z'}\Omega_H - i_{Z'}L_Z\Omega_H \\ &= 0 \end{aligned}$$

Therefore, $[Z, Z'] \in \ker \Omega_H$. We deduce that

$$[Y, \ker \Omega_H] = [X, \ker \Omega_H] + [\ker \Omega_H, \ker \Omega_H] \subset \ker \Omega_H.$$

Remark 12.3. Denote by $D(\mathfrak{X}^{(\Omega_H,\eta)}(M))$ the set of all the dynamical symmetries of $\mathfrak{X}^{(\Omega_H,\eta)}(M)$. It is easy to prove that if X and Y are two dynamical symmetries of $\mathfrak{X}^{(\Omega_H,\eta)}(M)$, then $[X, Y]$ is also a dynamical symmetry of $\mathfrak{X}^{(\Omega_H,\eta)}(M)$. Thus, $D(\mathfrak{X}^{(\Omega_H,\eta)}(M))$ is a Lie subalgebra of $\mathfrak{X}(M)$.

Definition 12.3. A Cartan symmetry of the precosymplectic system (M, Ω_H, η) is a vector field X on M such that

$$i_X\Omega_H = dG$$

where $G \in C^\infty(M)$.

Proposition 12.1. If X is a Cartan symmetry of the precosymplectic system (M, Ω_H, η), then X is a dynamical symmetry of $\mathfrak{X}^{(\Omega_H,\eta)}(M)$.

Proof. In fact, if $Z \in \ker \Omega_H$, we have

$$\begin{aligned} i_{[X,Z]}\Omega_H &= L_X i_Z\Omega_H - i_Z L_X\Omega_H \\ &= -i_Z d i_X\Omega_H = 0 \end{aligned}$$

Then, $[X, \ker \Omega_H] \subset \ker \Omega_H$ and therefore X is a dynamical symmetry of $\mathfrak{X}^{(\Omega_H,\eta)}(M)$. ∎

Remark 12.4. We denote by $C(\Omega_H, \eta)$ the set of all the Cartan symmetries of the precosymplectic system (M, Ω_H, η). If X and Y are Cartan symmetries such that

$$i_X\Omega_H = dG \qquad \text{and} \qquad i_Y\Omega_H = dG'$$

then $[X, Y]$ is a Cartan symmetry of (M, Ω_H, η). In fact,

$$\begin{aligned} i_{[X,Y]}\Omega_H &= L_X i_Y \Omega_H - i_Y L_X \Omega_H \\ &= L_X dG' - i_Y ddG \\ &= d(XG') \end{aligned}$$

The set $C(\Omega_H, \eta)$ is a Lie subalgebra of $\mathfrak{X}(M)$. Moreover, from Proposition 12.1 we deduce that $C(\Omega_H, \eta) \subset D(\mathfrak{X}^{(\Omega_H,\eta)}(M))$.

Theorem 12.1 (Noether Theorem). If a vector field X is a Cartan symmetry of the precosymplectic system (M, Ω_H, η), then G is a constant of the motion of $\mathfrak{X}^{(\Omega_H,\eta)}(M)$. Conversely, if G is a constant of the motion of $\mathfrak{X}^{(\Omega_H,\eta)}(M)$, the equation

$$i_X \Omega_H = dG$$

has a solution, and each solution is a Cartan symmetry of the precosymplectic system (M, Ω_H, η).

Proof. In fact, if X is a Cartan symmetry of the precosymplectic system (M, Ω_H, η), then, for all $Z \in \ker \Omega_H$, we obtain that

$$i_Z i_X \Omega_H = ZG = 0$$

Hence, G is a constant of the motion of $\mathfrak{X}^{(\Omega_H,\eta)}(M)$. Conversely, if G is a constant of the motion $\mathfrak{X}^{(\Omega_H,\eta)}(M)$, the equation

$$i_X \Omega_H = dG$$

has a solution because $(\ker \Omega_H)G = 0$, and therefore every solution is a Cartan symmetry of the system (M, Ω_H, η). ∎

2. *Second Case.* Ω_H does not have constant rank.

We know that (14) has a solution at the points x of M where $\text{rank}(\Omega_H)_x = 2r$. We define

$$M_2 = \{x \in M/\text{rank}(\Omega_H)_x = 2r\}$$

which we suppose to be a submanifold. On M_2, (14) has a solution for all $x \in M_2$, i.e., there exists a vector $v \in T_x M$ such that

$$i_v(\Omega_H)_x = 0 \qquad \text{and} \qquad i_v \eta_x = 1$$

There exists a vector field X on M_2 tangent to M_1 such that X satisfies (14). But, in general, X will not be tangent to M_2. Then we consider the submanifold M_3 where there exist solutions of (14) tangent to M_2. Following this process, we obtain a sequence of submanifolds M_l, $l = 1, \ldots$, called the l-ary constraint submanifolds. We can also define the submanifolds M_l as follows:

$$M_l = \{x \in M_{l-1}/\eta_x \in b_x(T_x M_{l-1})\}$$

where

$$b\colon\ TM \to T^*M$$
$$X \mapsto i_X\Omega_H + (i_X\eta)\eta$$

If the algorithm ends, we obtain the manifold M_f where the equations

$$(i_X\Omega_H = 0,\ i_X\eta = 1)_{/M_f}$$

have as a solution a vector field ξ on M_f. M_f is called the final constraint submanifold.

If we denote by j_f: $M_f \to M$ the embedding of M_f in M, then we can consider the precosymplectic system $(M_f, j_f^*\Omega_H, j_f^*\eta)$ and study the symmetries and constants of the motion for the precosymplectic system $\mathfrak{X}^{(j_f^*\Omega_H, j_f^*\eta)}(M_f)$ as in the first case.

We can apply these results to classify the symmetries and constants of the motion when we have a singular nonautonomous (or time-dependent) Lagrangian. In fact, if we suppose that L: R × $TQ \to$ R is a nonautonomous Lagrangian such that ω_{L_t} is a presymplectic 2-form of rank $2r$ on $\{t\} \times TQ \equiv TQ$, where $L_t(x) = L(t, x)$, $\forall x \in TQ$, then, if we define

$$\Theta_L = d_J L + E_L\, dt$$
$$\Omega_L = -d\Theta_L$$

we have that $2r \leq \text{rank}\ \Omega_L \leq 2r + 2$. The intrinsic motions equations are

$$i_X\Omega_L = 0, \qquad i_X\, dt = 1$$

As above, we can distinguish two cases, that is, rank $\Omega_L = 2r$ and otherwise. In addition, a finer analysis can be done by considering infinitesimal symmetries on the configuration space R × Q. The reader can obtain these results by taking into account those for the regular case (Prince, 1985; de Léon and Martín de Diego, 1995).

ACKNOWLEDGMENTS

This paper was partially supported by DGYCYT-SPAIN, Proyecto PB91-0142, and Departamento de Economía Aplicada Cuantitativa, UNED.

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